Quantum Langevin equation for a harmonic oscillator with memory-dependent damping

1988 J. Phys. A: Math. Gen. 21 1457

(http://iopscience.iop.org/0305-4470/21/6/020)

J. Phys. A: Math. Gen. 21 (1988) 1457-1465. Printed in the UK

Quantum Langevin equation for a harmonic oscillator with memory-dependent damping

R Chakrabarti and R Vasudevan

The Institute of Mathematical Sciences, Madras 600 113, India

Received 30 October 1987

Abstract. The heat-bath frequency distribution, necessary to maintain the canonical commutation relations for all time for a quantum mechanical oscillator with non-Markovian features like memory-dependent damping, is shown to satisfy certain constraint relations. An algorithm is given to explicitly find the heat-bath frequency distribution in terms of a series expansion for all processes where the timescale for the non-Markovian memory kernel is much smaller than the inverse of the strength of damping term. In the non-Markovian case, the heat-bath distribution function exhibits dependence on the system characteristics. The KMS periodicity conditions on the system Green functions as the system approaches equilibrium are established for the present case.

1. Introduction

Recently Streater (1982) investigated the quantum Langevin equation for a harmonic oscillator interacting weakly and linearly with a heat bath. Initially, at time $t=0$, the oscillator can be described by operators $q(0)$ and $p(0)$ acting on $L^2(\mathbb{R})$. The heat bath is assumed to be described by a single variable $\phi(t)$ and its momentum conjugate $\pi(t)$ acting on the Hilbert space Γ at times $t>0$. The dynamical variables $q(t)$ and $p(t)$ act on the full Hilbert space $L^2(\mathbb{R})\otimes\Gamma$. The key requirement is the validity of the canonical commutation relations for the dynamical variables $q(t)$ and $p(t)$ for all time $t>0$. This necessitates the quantum treatment of the heat bath, because otherwise the harmonic oscillator variables would violate the canonical commutation relations. Streater (1982) also adopted the physical restriction of describing the heat bath in terms of the positive frequency modes only. For a Markovian system, where the damping is instantaneous and can be described by a constant coefficient, Streater (1982) determined the frequency distribution $\rho(k)$ of the heat-bath oscillators. He also proved that the KMS periodicity conditions (Kubo 1957, Martin and Schwinger 1959), valid for the equilibrium field theory, can be dynamically obtained in the present problem by allowing $t\to\infty$.

In the present paper we investigate the problem of a quantum harmonic oscillator weakly coupled to a heat bath with damping term described by a memory-dependent kernel (MDK). In the classical case this problem was investigated by Kubo (1966) who obtained the fluctuation-dissipation relations. Our object here is to describe the heat bath for an arbitrary MDK, which may depend on more than one timescale. It is possible to explicitly determine the heat-bath frequency distribution $\rho(k)$ as a perturbation series, where the leading term reduces to the distribution for a Markovian process when the timescale(s) in the memory kernel $\tau\to 0$. In the limit where the memory scale

τ is small, the whole perturbation series can be determined in an order by order treatment. In addition, a complete set of consistency conditions for $\rho(k)$ for an arbitrary MDK can be established non-perturbatively. These are of the nature of dispersion relations. The KMS periodicity conditions can be dynamically established for an arbitrary MDK in the equilibrium limit. As a byproduct from the equal-time limit of the two-point Green function, one can get the expectation value of the particle density in a particular frequency mode.

The plan of this paper is as follows: we determine the heat-bath frequency distribution $\rho(k)$ in § 2. In § 3 we evaluate the two-point Green function for an arbitrary MDK in the non-equilibrium case and obtain the KMS periodicity conditions in the equilibrium limit. We conclude in § 4, drawing attention to physical results.

2. Heat-bath frequency distribution

The equations of motion of a quantum oscillator with an MDK $\gamma(t)$ in the presence of an additive heat-bath noise are

$$\begin{aligned}\dot{q}(t) &= \omega p(t) - \int_0^t \gamma(t-s)q(s)\,\mathrm{d}s + \phi(t)\\ \dot{p}(t) &= -\omega(t) - \int_0^t \gamma(t-s)p(s)\,\mathrm{d}s + \pi(t).\end{aligned} \tag{1}$$

If $a(t)$ and $\tilde{\alpha}(t)$ are the annihilation operators for the system and heat bath, respectively, we can rewrite the equation of motion as

$$\dot{a}(t) + \mathrm{i}\omega a(t) + \int_0^t \gamma(t-s)a(s)\,\mathrm{d}s = \tilde{\alpha}(t) \tag{2}$$

with $a(0) = a_0$.

The Fourier–Laplace transforms

$$\begin{aligned}a(t) &= \frac{1}{2\pi}\int_0^\infty a(k)\exp(-\mathrm{i}kt)\,\mathrm{d}k\\ \tilde{\alpha}(t) &= \frac{1}{2\pi}\int_0^\infty \rho(k)\alpha(k)\exp(-\mathrm{i}kt)\,\mathrm{d}k\end{aligned}$$

refer only to the positive-frequency mode of the system and the oscillator. The characteristics of the bath are described by the distribution function $\rho(k)$. The frequency modes satisfy the commutation relations

$$[\alpha(k), \alpha(k')^\dagger]_\mp = \delta(k-k'). \tag{3}$$

The upper sign refers to bosons and the lower sign refers to fermions, respectively.

The solution of (2) is given by

$$a(t) = \mu(t)a_0 + \int_0^t \mu(t-s)\tilde{\alpha}(s)\,\mathrm{d}s \tag{4}$$

where $\mu(t)$ is called the admittance function, whose transform $\mu(k)$ is given by

$$\mu(k) = \mathrm{i}[k - \omega + \mathrm{i}\tilde{\gamma}(k)]^{-1} \tag{5}$$

where $\tilde{\gamma}(k)$ is the transform of $\gamma(t)$. The admittance function is analytic in the upper half-plane as $\mu(t)=0$ for $t<0$.

A typical MDK with a strength γ and a timescale for memory τ is

$$\gamma(t-s)=(\gamma/\tau)\exp(-|t-s|/\tau)\theta(t-s). \tag{6}$$

In the limit $\gamma\tau\to 0$, the admittance kernel is

$$\mu(t)\simeq(1+\gamma\tau)\exp[-\mathrm{i}\omega(1+\gamma\tau)t-\gamma(1+\gamma\tau)] \\ -\gamma\tau\exp[-t/\tau+\gamma(1+\gamma\tau)t+\mathrm{i}\omega\gamma\tau t]. \tag{7}$$

The general form of $\mu(t)$ may be taken as

$$\mu(t)=\sum_i c_i\exp(-\mathrm{i}\omega_i t-\Gamma_i t) \tag{8}$$

with the constraints

$$\sum_i c_i=1 \qquad \Gamma_i>0.$$

This generalisation takes care of multiple timescales in the memory kernel. The index i goes over the poles of $\mu(k)$ in the lower half-plane, each pole relating to a timescale.

The requirement to be satisfied is

$$[a(t),a^{\dagger}(t)]_{\mp}=1 \qquad \text{for all } t>0 \tag{9}$$

where

$$[a_0,a_0^{\dagger}]_{\mp}=1. \tag{10}$$

We treat the case of bosons and fermions together. From (4), (9) and (10) we have

$$|\mu(t)|^2+\int_0^{\infty}\mathrm{d}k\,\rho^2(k)\left[\int_0^t\mathrm{d}s\,\mu(t-s)\exp(-\mathrm{i}ks)\right] \\ \times\left[\int_0^t\mathrm{d}s'\,\mu(t-s)^*\exp(\mathrm{i}ks')\right]=1. \tag{11}$$

This is the key equation for the subsequent development. As $t\to\infty$, the admittance kernel becomes damped:

$$|\mu(t)|\to 0.$$

Using this, we have from (11) a time-independent relation:

$$\int_0^{\infty}\mathrm{d}k\,\rho^2(k)|\mu(k)|^2=1. \tag{12}$$

The limit $t\to\infty$ is to be understood as t becoming large compared to the largest timescale in the memory kernel. The time-dependent part of (11) also leads to information about $\rho(k)$. To study it, we take the form of $\mu(t)$ as in (8), and (11) now takes the form

$$\sum_{i,j}c_ic_j^*\exp[-\mathrm{i}(\omega_i-\omega_j)t-(\Gamma_i+\Gamma_j)t] \\ +\int_0^{\infty}\mathrm{d}k\,\rho^2(k)\left[\sum_{i,j}c_ic_j^*\exp[-\mathrm{i}(\omega_i-\omega_j)t-(\Gamma_i+\Gamma_j)t]\right. \\ \times\left(\frac{\exp[\mathrm{i}(\omega_i-k)t+\Gamma_it]-1}{k-\omega_i+\mathrm{i}\Gamma_i}\right) \\ \left.\times\left(\frac{\exp[-\mathrm{i}(\omega_j-k)t+\Gamma_jt]-1}{k-\omega_j-\mathrm{i}\Gamma_j}\right)\right]=1. \tag{13}$$

We will first derive a set of constraints on $\rho(k)$ and later evaluate $\rho(k)$ as a series expansion.

The time-independent part of (13) is

$$\sum_{i,j} c_i c_j^* \int_0^\infty \mathrm{d}k\, \rho^2(k)[(k-\omega_i+\mathrm{i}\Gamma_i)(k-\omega_j-\mathrm{i}\Gamma_j)]^{-1}=1 \tag{14}$$

and the time-dependent part takes the form

$$\begin{aligned}\sum_{i,j} c_i c_j^* \bigg(& \exp[-i(\omega_i-\omega_j)t-(\Gamma_i+\Gamma_j)t] \\ & +\int_0^\infty \mathrm{d}k\, \rho^2(k)\frac{\exp[-\mathrm{i}(\omega_i-\omega_j)t-(\Gamma_i+\Gamma_j)t]}{(k-\omega_i+\mathrm{i}\Gamma_i)(k-\omega_j-\mathrm{i}\Gamma_j)} \\ & -\int_0^\infty \mathrm{d}k\, \rho^2(k)\frac{\exp[\mathrm{i}(k-\omega_i)t-\Gamma_i t]}{(k-\omega_i+\mathrm{i}\Gamma_i)(k-\omega_j-\mathrm{i}\Gamma_j)} \\ & -\int_0^\infty \mathrm{d}k\, \rho^2(k)\frac{\exp[-\mathrm{i}(k-\omega_j)t-\Gamma_j t]}{(k-\omega_j+\mathrm{i}\Gamma_j)(k-\omega_i-\mathrm{i}\Gamma_i)}\bigg)=0.\end{aligned} \tag{15}$$

In the case of a single pole in $\mu(k)$, (14) and (15) lead to the solution obtained by Streater (1982). We will adopt a slightly different route and obtain a set of dispersion relations obeyed by $\rho^2(k)$. Integrating the third and the fourth terms on the LHS of (15) and equating the coefficients of $\exp[-\mathrm{i}(\omega_i-\omega_j)t-(\Gamma_i+\Gamma_j)t]$ on both sides, we obtain a set of dispersion relations valid for all i and j,

$$\begin{aligned}1+\int_0^\infty & \mathrm{d}k\, \rho^2(k)[(k-\omega_i+\mathrm{i}\Gamma_i)(k-\omega_j-\mathrm{i}\Gamma_j)]^{-1} \\ & =\frac{2\pi i}{\omega_j-\omega_i+\mathrm{i}(\Gamma_i+\Gamma_j)}[\rho^2(\omega_i-\mathrm{i}\Gamma_i)+\rho^2(\omega_j+\mathrm{i}\Gamma_j)].\end{aligned} \tag{16}$$

An explicit solution for $\rho^2(k)$ is also possible in all cases where the MDK is not too far from a Markovian process. What we mean by this is best exemplified by considering (7) and (8). The kernel $\mu(t)$ in (7), when written in the form of (8), gives

$$\begin{array}{lll} c_1\simeq 1+\gamma\tau & \omega_1\simeq\omega(1+\gamma\tau) & \Gamma_1\simeq\gamma(1+\gamma\tau) \\ c_2\simeq -\gamma\tau & \omega_2\simeq -\omega\gamma\tau & \Gamma_2\simeq(1/\tau)(1+\gamma\tau)\end{array} \tag{17}$$

where in the limit $\gamma\tau\to 0$ we have $c_2\ll c_1$ and in the Markovian limit only c_1 is present. For all cases where all c_i $(i\neq 1)$ are $\mathrm{O}(\gamma\tau_A)$, where τ_A is a timescale in the memory, and only c_1 survives in limit $T_A\to 0$, we can explicitly solve for $\rho^2(k)$ as a perturbation series in $\gamma\tau_A$. For definiteness, we will work with the kernel in (7) and evaluate terms in $\rho^2(k)$ up to the first order in $\gamma\tau$.

For the rest of the present section, we will use the expansion

$$\rho^2(k)=D_0(k)+\gamma\tau D_1(k)+\mathrm{O}(\gamma\tau)^2. \tag{18}$$

For $k<0$, $\rho(k)$ vanishes at all orders of perturbation expansion, i.e. $D_l(k<0)=0$ for all l.

Using the expansion (18), (14) takes the form

$$\begin{aligned}\int_0^\infty & \mathrm{d}k[D_0(k)+\gamma\tau D_1(k)+\ldots]\{(1+2\gamma\tau)[(k-\omega_1)^2+\gamma_1^2]^{-1} \\ & -\gamma\tau[(k-\omega_1+\mathrm{i}\gamma_1)(k-\omega_2-\mathrm{i}\gamma_2)]^{-1} \\ & -\gamma\tau[(k-\omega_1-\mathrm{i}\gamma_1)(k-\omega_2+\mathrm{i}\gamma_2)]^{-1}\}=1.\end{aligned} \tag{19}$$

We equate the terms O(1) and terms O($\gamma\tau$) separately on both sides of (11), thus obtaining

$$\int_0^\infty \mathrm{d}k\, D_0(k)[(k-\omega_1)^2+\gamma_1^2]^{-1}=1 \tag{20}$$

and

$$\begin{aligned}\int_0^\infty \mathrm{d}k\, D_1(k)[(k-\omega_1)^2+\gamma_1^2]^{-1}&\\ =-2+\int_0^\infty \mathrm{d}k\, D_0(k)\{&[(k-\omega_1+\mathrm{i}\gamma_1)(k-\omega_2-\mathrm{i}\gamma_2)]^{-1}\\ &+[(k-\omega_1-\mathrm{i}\gamma_1)(k-\omega_2+\mathrm{i}\gamma_2)]^{-1}\}.\end{aligned} \tag{21}$$

Equation (15) on substitution in the expansion (18) gives

$$\begin{aligned}(1+2\gamma\tau)\exp(-2\Gamma_1 t)-2\gamma\tau\exp[-(\Gamma_1+\Gamma_2)t]\cos(\omega_1-\omega_2)t&\\ +\int_0^\infty \mathrm{d}k[D_0(k)+\gamma\tau D_1(k)][\![&(1+2\gamma\tau)[(k-\omega_1)^2+\Gamma_1^2]^{-1}\\ &\times[\exp(-2\Gamma_1 t)-2\exp(-\Gamma_1 t)\cos(k-\omega_1)t]\\ &-\gamma\tau[(k-\omega_1+\mathrm{i}\Gamma_1)(k-\omega_2-\mathrm{i}\Gamma_2)]^{-1}\{\exp[-\mathrm{i}(\omega_1-\omega_2)t-(\Gamma_1+\Gamma_2)t]\\ &-\exp[\mathrm{i}(k-\omega_1)t-\Gamma_1 t]-\exp[-\mathrm{i}(k-\omega_2)t-\Gamma_2 t]\}\\ &-\gamma\tau[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\{\exp[\mathrm{i}(\omega_1-\omega_2)t-(\Gamma_1+\Gamma_2)t]\\ &-\exp[-\mathrm{i}(k-\omega_1)t-\Gamma_1 t]-\exp[\mathrm{i}(k-\omega_2)t-\Gamma_2 t)]\}]\!]=0.\end{aligned} \tag{22}$$

Equating terms O(1) and O($\gamma\tau$) separately, we get

$$\int_0^\infty \mathrm{d}k\, D_0(k)[(k-\omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\omega_1)t=\exp(-\Gamma_1 t) \tag{23}$$

and

$$\begin{aligned}2\exp[-(\Gamma_1+\Gamma_2)t]\cos(\omega_1-\omega_2)t+\int_0^\infty \mathrm{d}k\, D_1(k)&\\ &\times[(k-\omega_1)^2+\Gamma_1^2]^{-1}[2\exp(-\Gamma_1 t)\cos(\omega_1-\omega_2)t-\exp(-2\Gamma_1 t)]\\ &+\int_0^\infty \mathrm{d}k\, D_0(k)[(k-\omega_1+\mathrm{i}\Gamma_1)(k-\omega_2-\mathrm{i}\Gamma_2)]^{-1}\\ &\times\{\exp[-\mathrm{i}(\omega_1-\omega_2)t-(\Gamma_1+\Gamma_2)t]-\exp[\mathrm{i}(k-\omega_1)t-\Gamma_1 t]\\ &-\exp[-\mathrm{i}(k-\omega_2)t-\Gamma_2 t]\}\\ &+\int_0^\infty \mathrm{d}k\, D_0(k)[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\\ &\times\{\exp[\mathrm{i}(\omega_1-\omega_2)t-(\Gamma_1+\Gamma_2)t]-\exp[-\mathrm{i}(k-\omega_1)t-\Gamma_1 t]\\ &-\exp[\mathrm{i}(k-\omega_2)t-\Gamma_2 t]\}=0.\end{aligned} \tag{24}$$

Equation (23) gives the cosine transform

$$\int_0^\infty \mathrm{d}k[D_0(k+\omega_1)+D_0(-k+\omega_1)](k^2+\gamma_1^2)^{-1}\cos kt=\exp(-\Gamma_1 t). \tag{25}$$

Since the cosine transform has a unique inverse, we must have

$$D_0(k+\omega_1)+D_0(-k+\omega_1)=2\Gamma_1/\pi. \tag{26}$$

The general solution of (26) was obtained by Streater. For an arbitrary measurable function $\Sigma_0(k)$ in the range $0\leqslant k\leqslant\omega_1$, satisfying $0\leqslant\Sigma_0(k)\leqslant 2\Gamma_1/\pi$ and $\Sigma_0(0)=\Gamma_1/\pi$, $D_0(k)$ is described by

$$D_0(k)=\begin{cases}0 & \text{for } k<0\\ \Sigma_0(\omega_1-k) & \text{for } 0\leqslant k<\omega_1\\ 2\Gamma_1/\pi-\Sigma_0(k-\omega_1) & \text{for } \omega_1\leqslant k<2\omega_1\\ 2\Gamma_1/\pi & \text{for } k\geqslant 2\omega_1.\end{cases} \tag{27}$$

Given $D_0(k)$, we can solve $D_1(k)$ by using (21) and (24); we get

$$\begin{aligned}\int_0^\infty & D_1(k)[(k-\omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\omega_1)t\,\mathrm{d}k\\
&=\exp(-\Gamma_1 t)\Big(-1+\tfrac{1}{2}\int_0^\infty \mathrm{d}k\,D_0(k)\{[(k-\omega_1+\mathrm{i}\Gamma_1)(k-\omega_2-\mathrm{i}\Gamma_2)]^{-1}\\
&\quad+[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\}\Big)\\
&\quad+\tfrac{1}{2}\exp[\mathrm{i}(\omega_1-\omega_2)t-\Gamma_2 t]\\
&\quad\times\Big(-1-\int_0^\infty \mathrm{d}k\,D_0(k)[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\\
&\quad+\frac{2\pi\mathrm{i}}{\omega_1-\omega_2+\mathrm{i}(\Gamma_1+\Gamma_2)}[D_0(\omega_1+\mathrm{i}\Gamma_1)+D_0(\omega_2-\mathrm{i}\Gamma_2)]\Big)\\
&\quad+\tfrac{1}{2}\exp[-\mathrm{i}(\omega_1-\omega_2)t-\Gamma_2 t]\\
&\quad\times\Big(-1-\int_0^\infty \mathrm{d}k\,D_0(k)[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\\
&\quad+\frac{2\pi\mathrm{i}}{\omega_2-\omega_1+\mathrm{i}(\Gamma_1+\Gamma_2)}[D_0(\omega_1-\mathrm{i}\Gamma_1)+D_0(\omega_2+\mathrm{i}\Gamma_2)]\Big).\end{aligned} \tag{28}$$

Notice that the terms $\sim\exp(-\Gamma_2 t)$ on the RHS of (28) vanish on account of (16), which is independent of the perturbation expansion used here. This reduces (28) to the following form:

$$\int_0^\infty \mathrm{d}k[(k-\omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\omega_1)t=F(\omega_1,\omega_2;\Gamma_1,\Gamma_2)\exp(-\Gamma_1 t) \tag{29}$$

where

$$\begin{aligned}F(\omega_1,\omega_2;\Gamma_1,\Gamma_2)&=-1+\tfrac{1}{2}\int_0^\infty \mathrm{d}k\,D_0(k)\\
&\quad\times\{[(k-\omega_1+\mathrm{i}\Gamma_1)(k-\omega_2-\mathrm{i}\Gamma_2)]^{-1}+[(k-\omega_1-\mathrm{i}\Gamma_1)(k-\omega_2+\mathrm{i}\Gamma_2)]^{-1}\}.\end{aligned}$$

Now the procedure for solving for $D_1(k)$ is exactly the same as $D_0(k)$ and we enlist

$$D_1(k)=\begin{cases}0 & \text{for } k<0\\ \Sigma_1(\omega_1-k) & \text{for } 0\leqslant k<\omega_1\\ (2\Gamma_1/\pi)F(\omega_1,\omega_2;\Gamma_1,\Gamma_2)-\Sigma_1(k-\omega_1) & \text{for } \omega_1\leqslant k<2\omega_1\\ (2\Gamma_1/\pi)F(\omega_1,\omega_2;\Gamma_1,\Gamma_2) & \text{for } k\geqslant 2\omega_1\end{cases} \tag{30}$$

where $\Sigma_1(k)$ is an arbitrary measurable function in the range $0\leqslant k<\omega_1$ with the value $\Sigma_1(0)=(\Gamma_1/\pi)F(\omega_1,\omega_2;\Gamma_1,\Gamma_2)$. This brings forward a novel feature of the non-Markovian processes. The solution for $D_0(k)$, the term which survives in the Markovian limit, is universal in nature. By this we mean that the heat-bath frequency distribution is independent of the system frequency ω except for the infrared cutoff for the range $k\leqslant\omega_1$. This can be understood that low-frequency modes of the heat bath cannot exchange energy with the system, which is at ground state most of the time. The strength of the first non-Markovian term $D_1(k)$, however, depends on the system through the function $F(\omega_1,\omega_2;\Gamma_1,\Gamma_2)$.

Thus we have shown that a systematic expansion scheme with the perturbation parameter $\gamma\tau$ can be developed where, starting from the lowest-order Markovian term, terms of any order in $\gamma\tau$ can be determined provided those of lower order are already known.

At each order of perturbation in $\gamma\tau$, an undetermined function (e.g. $\Sigma_0(k)$, $\Sigma_1(k)$, etc) has to be introduced to describe the distribution of heat-bath frequency modes. As explained by Streater (1982), they can be obtained by studying the lineshape of the spectrum.

3. Two-point Green function

The approach to thermal equilibrium of the system is studied by looking at the two-point Green function of the system. In particular, we establish the KMS (anti)periodicity condition for the two-point functions for (fermions) bosons in the equilibrium limit for an arbitrary non-Markovian process. In this section, we do not use the perturbation expansion with the expansion parameter $\gamma\tau$ used in § 2 for determination of $\rho^2(k)$.

The ensemble average for the heat-bath frequency modes is given by

$$\langle\alpha^+(k)\alpha(k')\rangle_\Omega=n(k)\delta(k-k') \tag{31}$$

where the occupation number per frequency mode is given by

$$n(k)=[\exp(\beta k)\mp 1]^{-1}. \tag{32}$$

The upper (lower) sign refers to bosons (fermions). Since we study the Green function for the system alone, we have to take an ensemble average over the heat bath. Using (4), we determine the system two-point function of two arbitrary times T and $T+t$,

$$\begin{aligned}\langle a^+(T)a(T+t)\rangle_\Omega=&\ \mu^*(T)\mu(T+t)\langle a_0^+a_0\rangle_\Omega\\ &+\int_0^\infty \mathrm{d}k\,\rho^2(k)n(k)\exp(-\mathrm{i}kt)\\ &\times\left(\int_0^T \mathrm{d}s\,\mu^*(s)\exp(-\mathrm{i}ks)\right)\left(\int_0^{T+t}\mathrm{d}s'\,\mu(s')\exp(\mathrm{i}ks')\right).\end{aligned} \tag{33}$$

As $T \to \infty$, the first term on the RHS vanishes and the factors in the parentheses converge to

$$\lim_{T\to\infty} \int_0^T \mathrm{d}s\, \mu^*(s) \exp(-\mathrm{i}ks) = \mu^*(k)$$

and

$$\lim_{T\to\infty} \int_0^{T+t} \mathrm{d}s\, \mu(s) \exp(\mathrm{i}ks) = \mu(k).$$

Using these results we obtain

$$G_0^<(t) \equiv \pm\langle a_\infty^\dagger a_\infty(t)\rangle_\Omega = \pm \int_0^\infty \mathrm{d}k\, \rho^2(k) n(k) |\mu(k)|^2 \exp(-\mathrm{i}kt). \tag{34}$$

Similarly, $\langle a(T+t)a^\dagger(T)\rangle_\Omega$ converges to

$$G_0^>(t) \equiv \langle a_\infty(t) a_\infty^\dagger\rangle_\Omega = \int_0^\infty \mathrm{d}k\, \rho^2(k)(1 \pm n(k)) |\mu(k)|^2 \exp(-\mathrm{i}kt). \tag{35}$$

At the $T \to \infty$ limit, the Green functions $G_0^<(t)$ and $G_0^>(t)$ are translation invariant. From (32), (34) and (35) it follows that

$$G_0^>(t - \mathrm{i}\beta) = \pm G_0^<(t) \tag{36}$$

which is the KMS condition. The present derivation is given for an arbitrary impedance kernel $\mu(k)$. The function $G_0^>(t)$ is defined in the strip $-\beta \leqslant \mathrm{Im}\, t \leqslant 0$ and $G_0^<(t)$ is defined in the strip $\beta \geqslant \mathrm{Im}\, t \geqslant 0$. For imaginary times t_{im} (where $t = -\mathrm{i}t_{\mathrm{im}}$) we can therefore define

$$G_0(t_{\mathrm{im}}) = \theta(t_{\mathrm{im}}) G_0^>(-\mathrm{i}t_{\mathrm{im}}) + \theta(-t_{\mathrm{im}}) G_0^<(-\mathrm{i}t_{\mathrm{im}}) \tag{37}$$

which is periodic in t_{im} with a period β. So far we have considered our system a damped harmonic oscillator. If the anharmonic terms are present, the full propagator G when it is convergent is given by Dyson's equation,

$$G = G_0 + G_0 S G \tag{38}$$

which leads to the periodicity of G as well. The self-energy function S can contain perturbative as well as non-perturbative contributions arising out of the anharmonic terms. So the statement about the periodicity of G does not depend on any expansion scheme. A similar argument applies to an arbitrary n-point function (Fetter and Walecka 1971). So we can conclude that a system with an MDK in the limit $T \to \infty$ reduces to the structure of an equilibrium thermal-field theory. This statement has to be understood in a qualified sense. Equation (38) refers to the imaginary-time Green functions. We make the Wick rotation along the imaginary time axis only after the real time $T \to \infty$, i.e. the harmonic oscillator Green function has already achieved equilibrium and consequently satisfies (36). It is assumed that S does not have any timescale comparable to T. Stated in a different way, the timescales arising out of the local anharmonic terms in the potential are smaller than T and therefore the time needed to come to equilibrium is governed by G_0 alone. This assumption may not be valid when tunnelling processes are present.

The occupation number operator for the damped harmonic system averaged over the heat bath can be determined explicitly for all time T:

$$N(T) \equiv \langle a^{+}(T)a(T)\rangle_{\Omega}$$

$$= |\mu(T)|^{2}\langle a_{0}^{+}a_{0}\rangle_{\Omega} + \int_{0}^{\infty} \mathrm{d}k\, \rho^{2}(k)n(k) \times\left(\int_{0}^{T} \mathrm{d}s\, \mu(s)\exp(-\mathrm{i}ks)\right)\left(\int_{0}^{T} \mathrm{d}s'\, \mu^{*}(s)\exp(\mathrm{i}ks')\right). \tag{39}$$

The equilibrium limit of the occupation number is

$$N \equiv \lim_{T\to\infty} N(T) = \int_{0}^{\infty} \mathrm{d}k\, n(k)\rho^{2}(k)|\mu(k)|^{2}. \tag{40}$$

4. Conclusion

The present formalism allows a very simple and straightforward description of a non-Markovian harmonic oscillator system in the presence of a heat bath. The non-Markovian processes are qualitatively different from the Markovian processes. For the Markovian processes the heat-bath mode density function $\rho(k)$ is universal in nature, i.e. independent of the equilibrating system apart from the infrared cutoff frequency. On the other hand, for the non-Markovian processes the heat-bath frequency distribution depends on the equilibrating system as well. The KMS periodicity conditions on the Green function dynamically appear in the equilibrium limit in much the same way for a non-Markovian damped system as for a Markovian damped system.

An alternative formalism of treating the heat-bath degrees of freedom fully dynamically was initiated by Caldeira and Leggett (1983). By using the path-integral mechanism, the heat-bath degrees of freedom were integrated out and an effective Lagrangian for the system was obtained. In this formalism the KMS periodicity of the Green function in the equilibrium limit has also been proved (Carlitz and Chakrabarti 1987). The distribution of the heat-bath frequency modes was obtained from the requirement that, in the semiclassical limit, the Langevin equation was satisfied. The formalism described in this paper treats the Langevin equation as a quantum mechanical operator equation and canonical (anti)commutation relations remain satisfied for all time; therefore it is fully quantum mechanical and not semiclassical in nature.

References

Caldeira A U and Leggett A J 1983 *Physica* **121A** 587
Carlitz R D and Chakrabarti R 1987 *Phys. Rev.* A **35** 3156
Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw-Hill)
Kubo R 1957 *J. Phys. Soc. Japan* **12** 571
—— 1966 *Rep. Prog. Phys.* **29** 225
Martin P C and Schwinger J 1959 *Phys. Rev.* **115** 1342
Streater R F 1982 *J. Phys. A: Math. Gen.* **15** 1477